# REFINED STRENGTH AND LIFE ANALYSIS OF FRP BEAMS

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## Abstract

The strength of FRP composites in civil infrastructure applications is often considered to be a secondary consideration, as design is driven by stiffness criteria. However, the long-term performance of FRP in such applications is not well understood. Initially high factors of safety can be reduced by ply-level degradation and changes in failure mode caused by environmental aging and fatigue loading. FRP structures under transverse loading frequently fail by delamination at material or geometric discontinuities. The stress state at these discontinuities is complex and difficult to predict, and the critical stresses are very sensitive to loading, fiber architecture, and ply-level changes with time/exposure. Furthermore, the out-of-plane strength at a delamination site may also be affected by environmental aging.

The present study considers several beam and sandwich type models to more accurately determine ply-level stresses that might lead to delamination at the ply level or at the web/flange interface of a structural beam. These models permit a relatively simple estimation of in-plane stresses which should be more accurate than those predicted using simple beam theory. They can capture such effects as load concentrations under point or patch loads, flange stresses due to web compression, and shear effects in both the web and flanges. Such models are advantageous as they permit analytic solutions, and they can be easily integrated into strength and life predictions which rely on iterative stress analyses and remaining strength calculations. The long-term objective of this work is to predict delamination failures in FRP thin-walled beams and sandwich beams under combined fatigue loading and environmental conditioning, using a simplified engineering approach. A more complete finite element model is also developed for verification.

# Introduction

Fiber-reinforced polymeric (FRP) composites are being considered for structural elements in bridge construction to increase durability and service life and to allow for faster installations. The design of FRP structures is typically stiffness-controlled, since large deflections can pose problems for overlay and connection durability. By satisfying the stiffness criterion, FRP structures will typically provide a high factor of safety on strength. For this reason, long-term durability of FRP as primary structural members has not been sufficiently addressed. However, the durability of FRP materials in critical, load-bearing structures under the influence of varied environmental factors is uncertain. Factors such as hygro-thermal cycles, UV exposure, and mechanical fatigue may have synergistic effects on the state of damage in a composite.

While many researchers have characterized the static performance of fiberglass structures, few have investigated the fatigue performance and environmental durability of such systems. Most durability work has been limited to coupon-level studies, and the development of life predictions for the structure based on the kinetics of damage mechanisms in coupon specimens has not been attempted. Furthermore, due to the presence of geometric and material discontinuities, failure often occurs by way of delamination at an interface, rather than fiber fracture at the ply level. Therefore, macro-level coupon studies may fail to predict the ultimate failure at the structural level. The objective of the present investigation is to fill this niche by developing a relatively simple analytical stress analysis for use in strength and life predictions of FRP structures.

The current study is motivated by the need to understand and predict the failure of a particular structural member that has been developed for the infrastructure market. Strongwell Corporation of Bristol, Virginia has developed a 91-cm (36-inch) deep pultruded double web beam (DWB) for use in bridge construction (Figure 1). The beam is a hybrid laminated composite, composed of both E-glass and carbon fibers in a vinyl ester resin. As a demonstration of the technology, a 12-m (40-ft) long single-span bridge in Marion, Virginia was rehabilitated in summer 2001 using eight double web beams. Prior to the rehabilitation, the beams to be implemented in the new structure were tested quasi-statically to obtain stiffness data. Failure testing of additional beams for the development of a design guide is nearing completion.

During the development of the 36-inch DWB, an 20-cm (8-inch) deep subscale prototype was studied and implemented in the Tom's Creek Bridge rehabilitation in 1997 [1]. Test results of the smaller section have been published in a structural design guide by Strongwell [2]. As tested in three- and four-point bending, the 8-inch DWB consistently fails by delamination within the compressive flange at an interface between carbon and glass fibers. The critical stress in this case is considered to be the normal, out-of-plane stress at the free edges. Furthermore, the delamination appears to initiate at the load points, suggesting that load concentrations have a significant effect on the critical stress. Therefore, an analytical model that can accurately predict ply-level stresses at the free edge and account for concentrated loading is required.

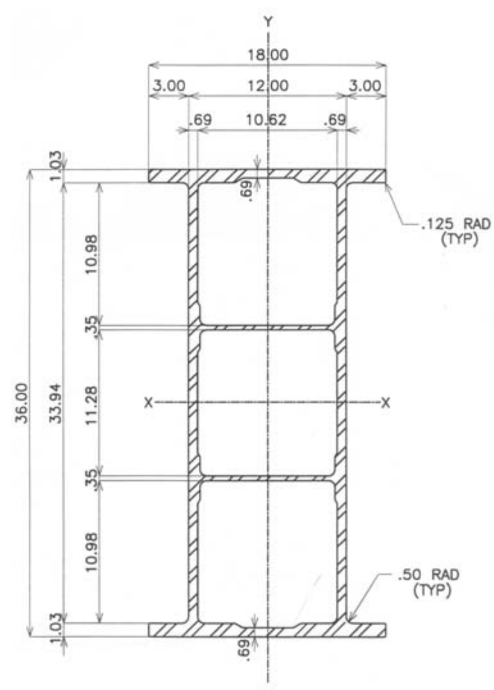


Figure 1. Strongwell's 36-inch DWB.

Senne [3] developed a simplified analytical model for the DWB to be used in strength and life predictions. This model uses a simple laminated beam theory to determine effective beam stiffnesses and then, assuming that the loading is pure bending, calculates the curvature of the flanges. In-plane stresses are determined using classical laminate theory (CLT), and the model applies a web "smearing" simplification to model the DWB as a rectangular beam, and then utilizes some type of boundary-layer solution to determine free-edge stresses. The two boundary-layer solutions attempted include the "Primitive Delamination Model" of Pipes and Pagano [4, 5] and the method by Kassapoglou and Lagace [6-8] based on assumed stress shapes that are optimized using the principle of minimum complementary energy.

The objective of the current study is to improve upon Senne's analysis by investigating the use of more sophisticated analytical global and local models, which can account for non-classical, higher-order effects. The current global analysis assumes pure bending loading and cannot capture localized effects due to stress concentrations and non-classical effects including web compressibility and shear warping. Furthermore, the local free-edge analysis contains some possibly constraining assumptions and restrictions.

## Double-Web Beam: Experimental Results

A number of three- and four-point bending tests at various span-to-depth ratios have been completed on both the 8-inch and 36-inch DWB sections [9]. In the case of the 36-inch DWB, five beams have been failed at spans of 18 m and 12 m (58 and 39 feet). Tests at a span of 5.5 m (18 ft) are underway. The test set-up includes two actuators located at roughly third points with 23-cm (9-inch) long steel-reinforced elastomeric bearing pads at both the supports and at the loading points (Figure 2) to reduce the severity of the load concentrations. The load from the hydraulic actuator cylinders are applied through a stout steel I-section which rests upon (and is welded to) a 2-inch thick steel plate. This assembly rests upon the bearing pads and is designed to maintain a uniform pressure distribution. The results of this testing indicate a linear relationship between shear capacity and span (see Figure 3), indicating that the failure is governed by shear capacity.

However, the observed failure mechanism is not shear failure in the web or at the web/flange interface, but rather delamination within the compressive flange at either a free edge or a ply drop-off (Figure 4). The delamination occurs at relatively low in-plane stresses, suggesting that the failure is not due to compression. The 8-inch DWB consistently fails by delamination in the top flange at the a carbon-glass interface, beginning in the vicinity of the loading patches and progressing outward in both directions along the primary beam axis. The 36-inch DWB appears to fail in a similar manner, although delamination may also initiate at the internal ply drop-off underneath the flange (refer to Figure 1). This drop-off is the result of glass fabrics in the inner half of each web that fold over in the pultrusion die to form the inner (all glass) portion of each flange.

Figure 2. Long-span testing of the 36-inch DWB.

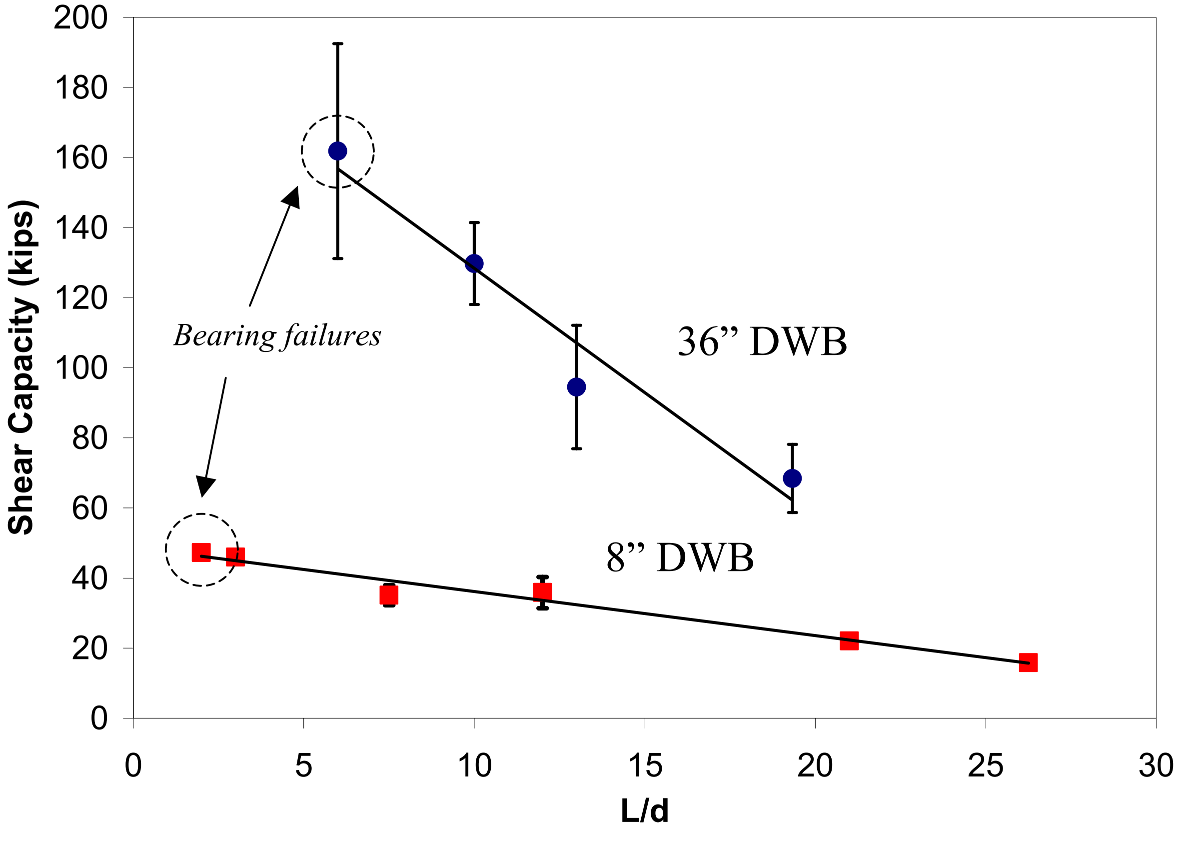


Figure 3. Shear capacity vs. span-to-depth ratio for the two different DWB sections. (Error bars representing one standard deviation are shown.)

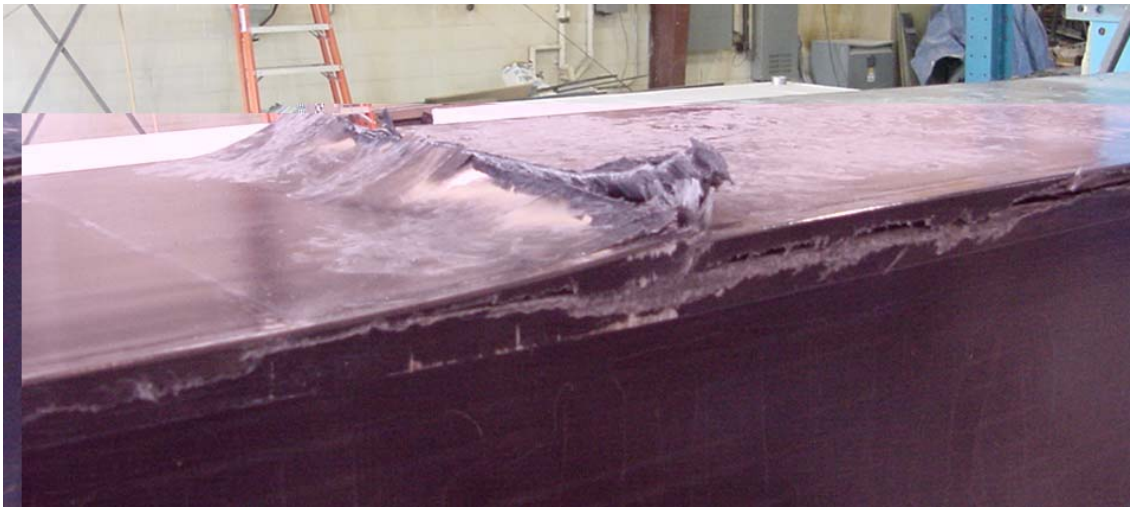

Figure 4. Delamination of the compressive flange of a 36-inch DWB under the loading patch.

Given the behavior of Figure 3, it seems that the critical interlaminar stresses are strongly influenced by the presence of transverse shear. Furthermore, because the delamination(s) initiate near loading points, it is reasonable to deduce that the concentrated loading has a localized effect on the critical stresses. This conclusion is supported by strain measurements taken during tests on the 36-inch DWB that indicate an change (decrease) in bending strain near the loading pads, relative to the mid-span measurements (Figure 5). This strain concentration may be caused by several factors including shear warping and transverse compressibility. Although these are mainly web effects, they will generate additional local bending curvatures in the flanges and cause a strain concentration.

Shear strain measurements in the vicinity of the load patches and the supports have been performed to assess shear warping behavior. A comparison of the shear strain distribution beneath one of the load patches for two different spans is given in Figure 6. The measurements were taken along the centerline of the beam. In both cases, the pad width was 23 cm (9 inches), and the results indicate that a length equal to about one pad's width beyond the edge of the pad is required for the shear strain to reach its far-field value. Over this transfer region, the flange is carrying part of the shear load and experiencing additional curvature, as evidenced by Figure 5. This effect was addressed in the early sandwich beam literature by Allen [10] and Plantema [11]. Similar effects caused by symmetry conditions (e.g. under three-point loading) have been addressed by Dufort et al. [12, 13].

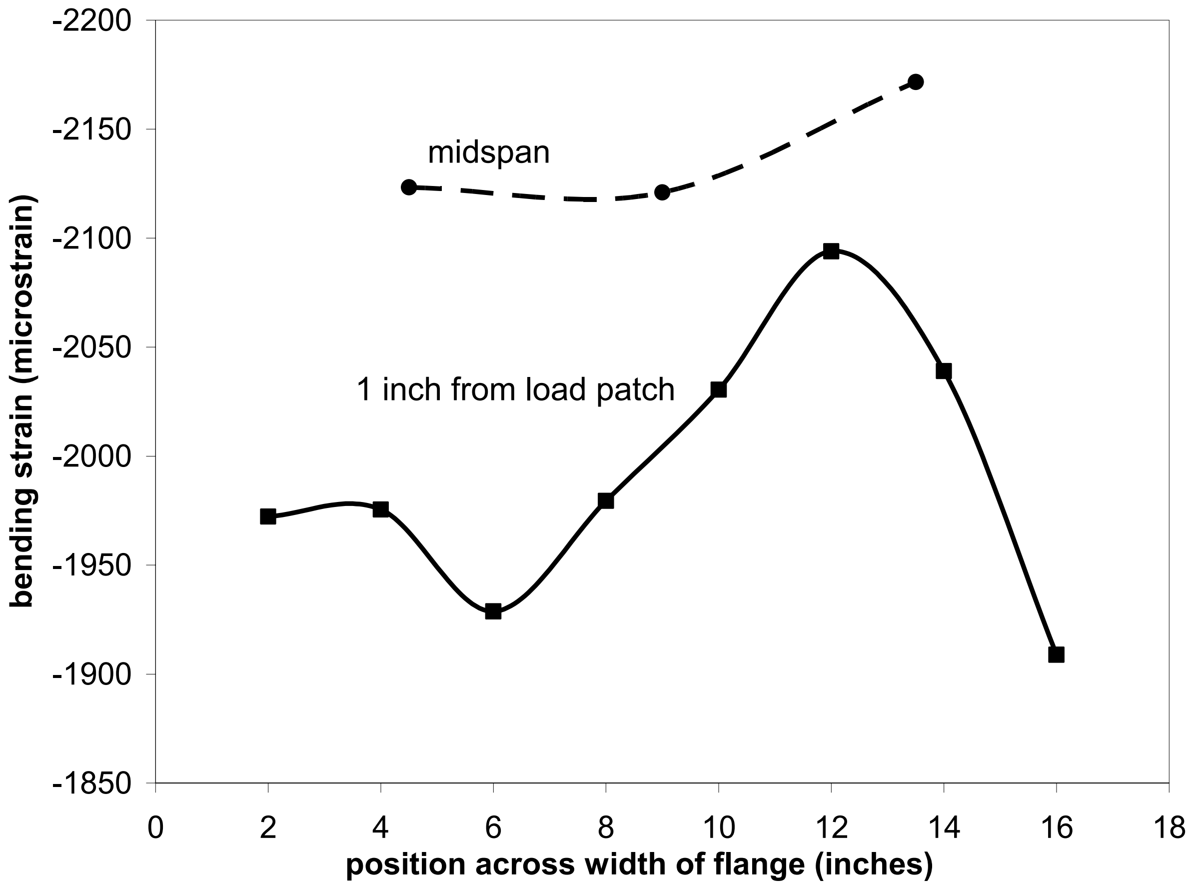


Figure 5. Top flange bending strains at mid-span and 2.5 cm (one inch) outside of the load patch.

Another effect that is typically neglected in beam models is transverse flexibility. Structures with a low transverse rigidity may experience significant transverse compressive strain. This behavior is common in sandwich panels that employ a soft, flexible core, and it can lead to crushing of the core. Compressible-core models have been developed to address this issue, and the results suggest that compression under the load points (and above the supports) will affect the stress state in the face sheets by inducing additional curvature [14, 15]. For thin-walled FRP beams, the relatively low compressive stiffness of the web may also permit an appreciable amount of compressive deformation. In order to assess the amount of web compression in the 36-inch DWB, strain measurements in the transverse direction were taken under and near one load patch during a 12 m (39 ft) span test. The results are given in Figure 7 for a total load of 267 kN (60 kips). The data indicate significant compressive strains directly under the load. Note that the maximum measured strain is 35% of the maximum bending strain on the flange. It is also noted that the strain measured closest to the flange appears to change sign just outside the load patch. This type of behavior is similar to that of a beam on an elastic foundation. These observations suggest that a strict beam model which does not permit transverse strain may not be appropriate for modeling some FRP thin-walled beams.

Consideration of these local effects is important not only in comparing analytical predictions with experimental data, but also in designing for real applications. Concentrated loading may occur under normal service conditions in some applications. Furthermore, the transfer of load from the beam to the supports occurs over a short length and therefore represents a load concentration. In the case of a uniformly distributed load, the stresses will be largest in the bottom flange at the support.

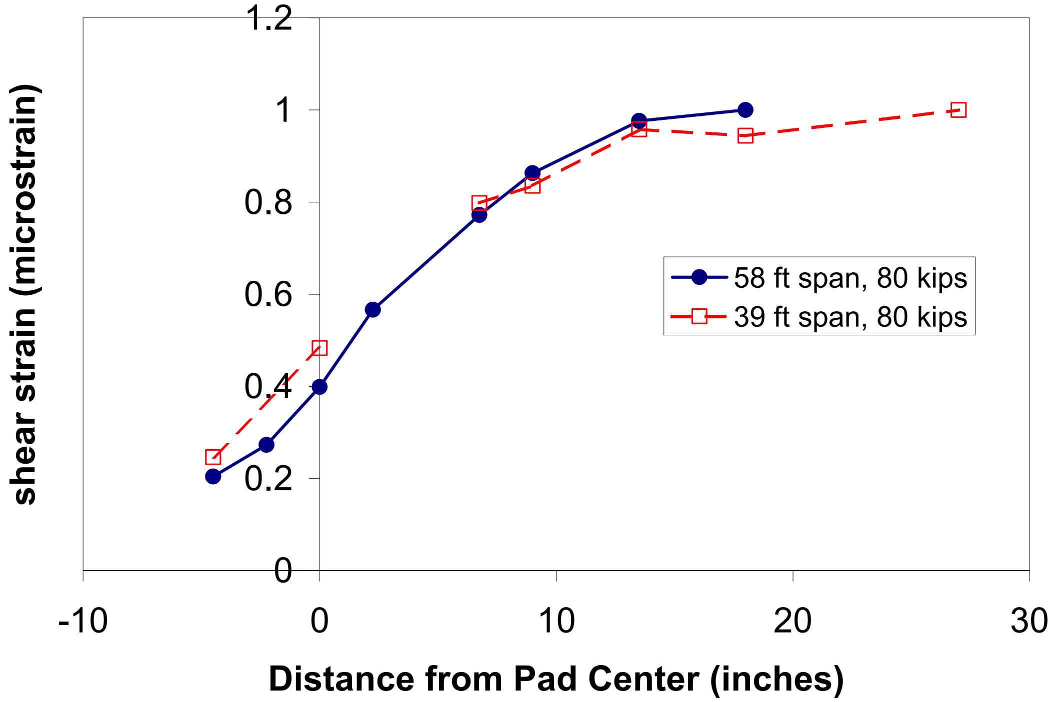


Figure 6. Normalized shear strain distribution along the centerline of the 36 inch DWB under one of the load patches during two four-point bending tests: 1) 18 m (58 ft) span and 2) 12 m (39 ft) span (total load =356 kN or 80 kips).

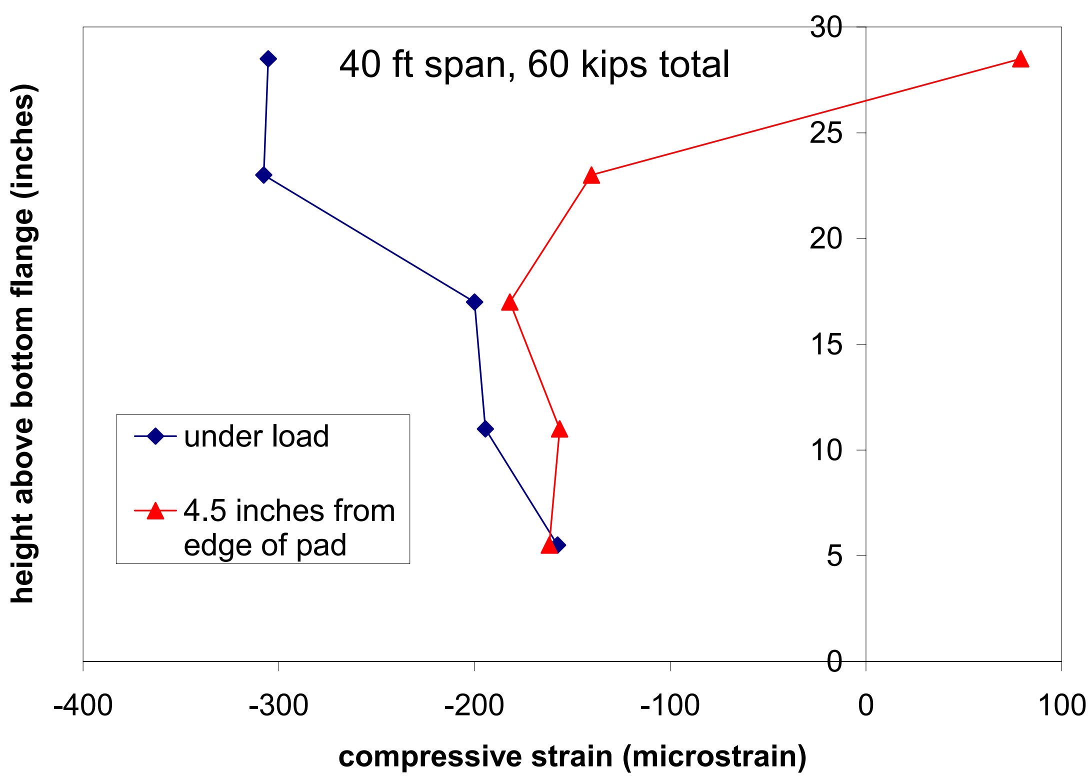


Figure 7. Transverse compressive strains through the depth of the web under and near the load patch.

## Stress Analysis

### *Global Models*

The experimental results indicate that the concentrated transverse load patches represent stress concentrations. In order to develop a stress analysis for use in strength and life predictions, a global-local type approach using analytical sub-models is considered. The approach by Senne [3] utilized a laminated beam theory to determine effective stiffness values for each panel comprising the section. The moment at any section was determined from ordinary beam theory, and the stiffness contribution of the flanges was used to then determine the curvature of the flanges. The loading was assumed to be pure bending, so that any load concentrations at the load points were neglected. Furthermore, the effect of transverse shear on the flanges was neglected.

An analytical thin-walled laminated beam theory which could capture higher order effects (including accurate ply-level stresses) is desired, but no such theory is available. For bridge applications, secondary bending and torsion are considered to be minor as compared to primary bending, so we may consider a simpler approach than what is often developed in the literature for rotors and other complex loaded structures. One possible approach is to model the beam as a sandwich structure, that is, two thick face sheets (flanges) separated by a less stiff core (web). Frostig's "higher-order sandwich theory" [15] and the elasticity solution for rectangular composites or sandwich plates by Pagano [16] and Srinivas and Rao [17] are considered. In the higher-order sandwich theory, the faces are modeled as beams and the core is considered to be an elastic medium, represented by the 2-D equations of equilibrium. Frostig assumes that the core carries no axial stress (the so-called "weak core" assumption), and the solution to the bending problem under either a uniform or central point load is found using an energy approach. The in-plane stresses in the face sheets and the transverse stresses at the face/core interface are shown to be consistent with the elasticity solution (see below) and finite element results.

For the more general case of a layered structure, Pagano's elasticity theory is more appropriate. Each layer in the structure is modeled as an elastic continuum, and the 3-D governing equations of equilibrium are expressed for each layer. Displacement and stress continuity are imposed as boundary conditions for each layer, and the solution is found using a Fourier expansion for the loading and by solving the system of constraint equations. For a sandwich panel or beam, the web is represented using its effective ply properties, and the core layer can be modeled as weak or "strong" (i.e. carrying axial stress). For a panel composed of $N$ layers, the solution requires solving a set of $4N$ linear coupled equations for each term of the loading expansion, and the solution becomes computationally intensive for beams with many layers.

Compared to simple beam theory, both the higher-order and elasticity models allow different boundary conditions on the two faces so that the effect of applied loads and supports can more accurately be represented. Furthermore, the solutions should yield more accurate through-the-thickness stresses since no kinematic assumptions about the displacement field are made *a priori*. For these reasons, these two models are considered for global analysis of the DWB. This requires that the web be model as a continuous core. The web panels of the DWB roughly approximate a weak core, since most of the bending stiffness is given by the flanges (96% for the 8-inch DWB and 88% for the 36-inch DWB) due to the use of carbon fiber in the outer portion of the flanges and mostly off-axis glass fiber in the web sections. The web is replaced by a single effective ply, and the lamina stiffness matrix of the new ply is calculated using the extensional stiffness matrix [A] from classical laminate theory.

Because the models of Frostig and Pagano are developed for solid, rectangular cross-sections, applying the models to thin-walled beams like the DWB neglects stress-free boundary conditions on the inner surfaces of the flanges. Additionally, the nature of the web-flange interface, including shear lag across the flange width, cannot be accurately represented. For now, the effect of shear lag is neglected and the variation of stress quantities in the y-direction (along the width of the flange) is assumed to be negligible. However, this validity of this assumption, as well as the effect of violating the stress-free boundary condition at the inner flange surfaces, is unclear and is to be investigated in future work (including finite element analysis). The elasticity solution also requires that each ply be simply-supported at the ends. This assumption is incorrect since only the bottom surface is constrained as such. Nevertheless, for the usual situation of concentrated loads, the stress state at the supports is not important, and the results far away from the supports should be fairly accurate.

***Local Models***

Senne [3] determined the curvature of the flange from simple beam theory and the ply-level, in-plane stresses using Classical Laminate Theory (CLT). These stresses were then utilized as input in one of the local boundary layer models [4-8]. This approach can be applied using one of the more sophisticated global models discussed in the previous section, if we recognize that the curvature will now vary with position along the length of the beam. Specifically, the curvature should be greatest near the load patches. Pagano's elasticity solution or Swanson's extension of Frostig's higher order sandwich model to the case of orthotropic faces will provide in-plane stresses directly [18].

Senne first investigated the "Primitive Delamination Model" by Pipes and Pagano [4, 5] for the local, free-edge analysis. The first model assumes a linearized form of the distribution of the out-of-plane normal stress $\sigma_z$ along the width of the boundary region, a distance equal to the laminate thickness. The moment on the laminate caused by $\sigma_z$ is balanced by that caused by the in-plane transverse stress $\sigma_y$. Finite difference solutions were later developed for the case of axial strain [16, 19], but these methods were not practical for thick, general laminates. Pipes and Pagano [20] then developed an approximate elasticity solution for angle-ply laminates in which the solution is found using separation of variables. These solutions indicated the existence of a singularity in the stress field at the free edge of a ply interface, as modeled by homogeneous orthotropic laminae.

Wang and Choi [21, 22] later used Lekhnitskii's stress potential and anisotropic elasticity to characterize the order of the singularities. The approach requires solving an eigenvalue problem, and traction continuity is accomplished using collocation at every ply. The procedure is tedious and therefore difficult for thick laminates. Later attempts to solve the problem using finite difference or finite element solutions have confirmed this result, requiring very refined meshes to approximate the order of the singularity [23-25]. For real materials, the plies are heterogeneous, and a singularity does not actually exist. Furthermore, most out-of-plane strength criteria depend upon an average stress calculated across the width of the boundary region [26, 27]. Thus, many researchers have developed analytical approaches that provide efficient, approximate estimates of the free edge stresses.

Kassapoglou and Lagace [6-8] developed an analytical solution using assumed stress shapes of a separable form that are then optimized using the principle of minimum complementary energy. This method is efficient for thick laminates, but it does not accurately capture the mismatch in ply properties through the thickness of the laminate. Therefore, the stress-free boundary conditions on the free edge are only met on a point-wise basis, not continuously through the depth. The original formulation was also limited to the case of symmetric laminates and in-plane loading (uniaxial deformation and pure bending). The method was later extended to non-symmetric laminates and combined loads [8, 28], and thus provides sufficient generality for many applications including transverse loading. More recent work has aimed to improve accuracy of these assumed-stress function approaches by using the Kantorovich method with assumed stress functions in the out-of-plane direction and unknown functions in the in-plane direction [29], or by using the Extended Kantorovich method to iteratively converge upon the exact in-plane and out-of-plane functions [30-32].

Despite the refinements in the approximate assumed-stress solutions, all assume that there are no variations in the primary beam axis direction. Therefore, local stress concentrations due to loads or supports cannot be captured. To overcome this limitation, Kim and Atluri [33] have derived a solution similar to previous assumed-stress approaches, which includes the longitudinal degrees of freedom in the stress distributions. The results of this approach suggest that the free edge stress shapes under out-of-plane loading can be very different than those predicted for in-plane loading.

The use of these free-edge solutions for use in the present local model is to be considered in future work. The importance of including the longitudinal degrees of freedom will be investigated. The 36-inch DWB presents an additional challenge not present in Senne's study of the 8-inch section, as the ply drop-offs on the inner flange surfaces present a second initiation site for delamination. Thus, the boundary layer solution must be applied separately to both the free edge and ply drop-off in a complete strength prediction.

***Finite Element Verifications***

In order to guide the development of an analytical model, a finite element model is being developed in ANSYS®. The panels composing the DWB are meshed using orthotropic, layered shell elements. The bearing pads are represented using rectangular volumes with soft, elastomeric properties, and the steel loading plates are also represented as rectangular volumes. The plate and pad are simply bonded together using the "glue" feature, and the pad is bonded to the flange. Uniform pressure tractions are applied to the steel plates, and the supports are modeled using fixity of displacement across the width of the bottom flange along a line representing the centerline of the support pad. The mesh is sufficiently refined in the beam axis direction to capture any stress concentrations. The global results are then applied to a sub-model constructed by cutting out a piece of the flange around one load patch. The nodal displacements and forces at the slices are applied as boundary conditions in a subsequent solution. In this solution, the flange is re-meshed using individual SOLID95 elements to represent each ply separately. Results of the FE model will be presented in future work. One focus of the FE modeling is to assess the affect of the flange-web interface on the free edge stresses. Another key focus is to identify the precise location where delamination initiates. Finally, we will assess the influence on the critical stresses of how the load patch is modeled.

## Summary

The results of full-scale tests on both the 8-inch and 36-inch double-web beams have been presented. The failure mechanism appears to be delamination at the outer free edges of the compressive flange or the inner ply drop-offs on the compressive flange, in the case of the 36-inch DWB. The application of several beam and sandwich models in the global stress analysis has been discussed, and the challenges of capturing local effects have been presented. These same considerations will impact the local free-edge analysis as well. The various boundary layer solutions for combined loading will be investigated for use in the local analysis, and the resulting stress analysis will be verified using finite elements.

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